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CASE STUDY

An analytical proposal for solution of Bessel Equation applied on heat transfer for adiabatic finds

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ABSTRACT
In this paper we present the mathematical proof of the solution of the Bessel equation applied to heat transfer for adiabatic extended surfaces. Although the final expression is of common use in engineering such as chemical, mechanical and related engineering, its obtaining from the analysis of the Bessel solution is not easy to find. The solution was obtained by the typical Frobenius Series method.

Keywords: Extended surfaces; Frobenius Series Method; Gamma Function; One-dimension conduction.

Una solución de la Ecuación de Bessel aplicada para la transferencia de calor en aletas adiabáticas

RESUMEN
En este estudio de caso se presenta la demostración matemática de la solución de la ecuación de Bessel aplicada a la transferencia de calor para superficies extendidas adiabáticas. Aunque la expresión final es de uso común en ingenierías como la química, mecánica y afines, su obtención a partir del análisis de la solución de Bessel no es fácil de encontrar. La solución se ha obtenido por el método típico de la serie de Frobenius.

Palabras clave: Conducción unidimensional, Función Gamma, Series de Frobenius, Superficies Extendidas.

ACCESO ABIERTO

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1. Introduction

The next equation is the second order differential equation for thermal profile of adiabatic find [1]:

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}-\frac{2h}{kt}(T-T_\infty)=0 \qquad (1)$$

with h is the local convection coefficient, k is the thermal conductivity, t is the thickness, $T_\infty$is the fluid temperature, r is the radius coordinate y T is the temperature of the system. The Eq. (1) is a special form of Bessel Equation, which can be writing as parametric form [1]:

$$\frac{\partial^2 \theta}{\partial \lambda^2}+\frac{1}{\lambda}\frac{\partial \theta}{\partial \lambda}-m^2\theta=0 \qquad (2)$$

with:

$\theta = T - T_\infty$

$r = \lambda$

$m^2 = \frac{2h}{kt}$

The boundary conditions of this equation are:

$$\lambda = \lambda_1 \rightarrow \theta = \theta_B$$

$$\lambda = \lambda_2 \rightarrow \frac{\partial \theta}{\partial \lambda}|_{\lambda=\lambda_2} = 0$$

The general solution of the modified Bessel equation using a Frobenius is:

$$\theta(\lambda) = C_1 I_0(m\lambda) + C_2 K_0(m\lambda) \qquad (3)$$

where I_0 y K_0 are modified Bessel function of zero order of first and second class, respectively, and C_1 y C_2 are integration constants. The evaluation of the boundary conditions permits the obtention of the Eq.(4) in function of I_0 y K_0.

$$\frac{\theta(\lambda)}{\theta_B}=\frac{I_0(m\lambda)K_1(m\lambda_2)+K_0(m\lambda)I_1(m\lambda_2)}{I_0(m\lambda_1)K_1(m\lambda_2)+K_0(m\lambda)I_1(m\lambda_2)} \qquad (4)$$

where $I_1(m\lambda)=\frac{d[I_0(m\lambda)]}{d\lambda}$ and $K_1(m\lambda)=-\frac{d[K_0(m\lambda)]}{d\lambda}$ are modified Bessel function of first and second order, for first and second class, respectively.

2. Demonstration (Results)

Modified Bessel differential equation has a next form [2]:

$$x^2\frac{d^2y}{dy^2}+x\frac{dy}{dx}+(m^2x^2-v^2)y=0 \quad (5)$$

Comparing the Eq. (5) with Eq. (2), we noted that Eq. (2) has a modified form of Bessel Equation.

The typical solution of the Eq. (5) is:

$$y(x)=C_1I_v(mx)+C_2K_v(mx) \quad (6)$$

Using the parametric notation in function of θ and λ:

$$\theta(\lambda)=C_1I_v(m\lambda)+C_2K_v(m\lambda) \quad (7)$$

If $v = 0$, so it obtains the modified Bessel of zero order of first and second class:

$$\theta(\lambda)=C_1I_0(m\lambda)+C_2K_0(m\lambda) \quad (8)$$

where $I_0(m\lambda)$ is a Bessel function of order zero of the first class and $K_0(m\lambda)$ is a Bessel function of order zero of second class.

Applying the Frobenius Method, the following solution is proposed for the second order equation [3]:

$$\theta=\sum_{n=0}^{\infty}C_n\lambda^{n+\alpha} \quad (9)$$

Now, the first and second derivatives are calculated:

$$\frac{\partial\theta}{\partial\lambda}=\sum_{n=0}^{\infty}(n+\alpha)C_n\lambda^{n+\alpha-1} \quad (10)$$

$$\frac{\partial^2\theta}{\partial\lambda^2}=\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)C_n\lambda^{n+\alpha-2} \quad (11)$$

Replacing Equation (10) and Eq. (11) in the differential Eq. (1) we have:

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)C_n\lambda^{n+\alpha-2}+\frac{1}{\lambda}\sum_{n=0}^{\infty}(n+\alpha)C_n\lambda^{n+\alpha-1}-m^2\sum_{n=0}^{\infty}C_n\lambda^{n+\alpha}=0$$

Factoring λ^{α} parameter:

$$\lambda^{\alpha}[\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)C_n\lambda^{n-2}+\frac{1}{\lambda}\sum_{n=0}^{\infty}(n+\alpha)C_n\lambda^{n-1}-m^2\sum_{n=0}^{\infty}C_n\lambda^{n}]=0$$

$$\lambda^{\alpha}[\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)C_n\lambda^{n-2}+\sum_{n=0}^{\infty}(n+\alpha)C_n\lambda^{n-2}-m^2\sum_{n=0}^{\infty}C_n\lambda^n]=0$$

$$\lambda^{\alpha}[\sum_{n=0}^{\infty}((n+\alpha)(n+\alpha-1)+(n+\alpha))C_n\lambda^{n-2}-m^2\sum_{n=0}^{\infty}C_n\lambda^n]=0$$

$$\lambda^{\alpha}[\sum_{n=0}^{\infty}((n+\alpha)(n+\alpha))C_n\lambda^{n-2}-m^2\sum_{n=0}^{\infty}C_n\lambda^n]=0$$

$$\lambda^{\alpha}[\sum_{n=0}^{\infty}(n+\alpha)^2C_n\lambda^{n-2}-m^2\sum_{n=0}^{\infty}C_n\lambda^n]=0$$

Expanding the first series in $n = 0$:

$$\lambda^{\alpha}[\alpha^2C_0\lambda^{-2}+\sum_{n=1}^{\infty}(n+\alpha)^2C_n\lambda^{n-2}-m^2\sum_{n=0}^{\infty}C_n\lambda^n]=0$$

Now, we have a first result $\alpha^2C_0\lambda^{\alpha-2}=0$, with logical restrictions $C_0\neq 0$ y $\lambda^{\alpha-2}\neq 0$ then:

$$\sum_{n=1}^{\infty}(n+0)^2C_n\lambda^{n-2}-m^2\sum_{n=0}^{\infty}C_n\lambda^n=0$$

$$\sum_{n=1}^{\infty}n^2C_n\lambda^{n-2}-m^2\sum_{n=0}^{\infty}C_n\lambda^n=0$$

In order to factor the summations it is necessary that the limits of the series as the powers of the function are symmetric. We expand the first series by $n = 1$:

$$C_1\lambda^{-1}+\sum_{n=2}^{\infty}n^2C_n\lambda^{n-2}-m^2\sum_{n=0}^{\infty}C_n\lambda^n=0$$

By doing $n = k + 2$ for the first series, and $n = l$ for the second series:

$$C_1\lambda^{-1}+\sum_{k=0}^{\infty}(k+2)^2C_{k+2}\lambda^k-m^2\sum_{l=0}^{\infty}C_l\lambda^l=0$$

It was obtained that the series and their powers are symmetrical, therefore $k = l$

$$C_1\lambda^{-1}+\sum_{k=0}^{\infty}(k+2)^2C_{k+2}\lambda^k-m^2\sum_{k=0}^{\infty}C_k\lambda^k=0$$

The sum is factored and ordered:

$$C_1\lambda^{-1}+\sum_{k=0}^{\infty}[(k+2)^2C_{k+2}-m^2C_k]\lambda^k=0$$

A second important result is obtained, $C_1\lambda^{-1}=0$, pero $\lambda^{-1}\neq 0$, so $C_1=0$, therefore:

$$\sum_{k=0}^{\infty}[(k+2)^2C_{k+2}-m^2C_k]\lambda^k=0$$

with condition:

$$(k+2)^2C_{k+2}-m^2C_k=0$$

It is explicitly expressed C_{k+2} as:

$$C_{k+2}=\frac{C_k}{(k+2)^2}m^2 \qquad (12)$$

A generalized recurrence relationship is then obtained for C_{k+2}:

$k=0$: $C_2=\frac{C_0}{(2)^2}m^2$

$k=1$: $C_3=\frac{C_1}{(3)^2}m^2=0\ \rightarrow C_1=0$

$k=2$: $C_4=\frac{C_2}{(4)^2}m^2\ \rightarrow C_4=\frac{1}{(4)^2}\frac{C_0}{(2)^2}m^4$

$k=3$: $C_5=\frac{C_3}{(5)^2}m^2=0\ \rightarrow C_3=0$

$k=4$: $C_6=\frac{C_4}{(6)^2}m^2\ \rightarrow C_6=\frac{1}{(6)^2}\frac{C_0}{(4)^2(2)^2}m^6\rightarrow\frac{1}{(6)^2}\frac{C_0}{(2)^6}m^6$

$k=5$: $C_7=\frac{C_5}{(7)^2}m^2=0\ \rightarrow C_5=0$

$k=6$: $C_8=\frac{C_6}{(8)^2}m^2\rightarrow C_8=\frac{1}{(8)^2}\frac{C_0}{(6)^2(2)^2}m^{12}\rightarrow\frac{1}{(6)^2}\frac{C_0}{(2)^6}m^{12}$

From the above expressions it can be concluded that the odd terms are zero, while the pairs are non-zero, now be it *2n = k + 2*, then *k = 2n – 2*, replacing in Eq. (12):

$$C_{2n}=\frac{C_{2n-2}}{(2n)^2}m^2$$

developing the expression:

$n = 1: \quad C_2 = \frac{C_0}{(2)^2} m^2$

$n = 2: \quad C_4 = \frac{C_2}{(4)^2} m^2 \rightarrow \frac{1}{(4)^2} \frac{C_0}{(2)^2} m^4$

$n = 3: \quad C_6 = \frac{C_4}{(6)^2} m^2 \rightarrow C_6 = \frac{1}{(6)^2} \frac{1}{(4)^2} \frac{C_0}{(2)^2} m^6 \rightarrow C_6 = \frac{1}{(6)^2} \frac{C_0}{(2)^6} m^6$

Generalizing, in terms of C_0:

$$C_{2n} = \frac{C_0}{(n!)^2} \left(\frac{m}{2}\right)^{2n} \tag{13}$$

Replacing in Eq. (9) and according to the results obtained for $\alpha = 0$:

$$\theta = \sum_{n=0}^{\infty} \frac{C_0}{(n!)^2} \left(\frac{m}{2}\right)^2 \lambda^{2n} \tag{14}$$

$$\theta = C_0 \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{m\lambda}{2}\right)^{2n} \tag{15}$$

The above expression corresponds to a modified Bessel function of order zero and of first class [4]:

$$I_0(m\lambda) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{m\lambda}{2}\right)^{2n} \tag{16}$$

By replacing in (9):

$$\theta = C_0 I_0(m\lambda) \tag{17}$$

We now determine the second-class modified Bessel function, which by definition is expressed as follows [4]:

$$K_v(x) = \frac{\pi(I_{-v}(x) - I_v(x))}{2\sin(\pi v)} \tag{18}$$

Applying limit: $v \rightarrow \mu$

$$K_\mu(x) = \lim_{v \to \mu} K_v(x)$$

Using L'Hopital for the limit of $K_n(x)$:

$$K_\mu(x) = \lim_{v \to \mu} K_v(x) = \lim_{v \to \mu} \frac{\frac{\partial}{\partial v}[\pi(I_{-v}(x) - I_v(x))]}{\frac{\partial}{\partial v}(2\sin(\pi v))}$$

$$= \lim_{v\to\mu} \frac{\pi\left[\frac{\partial}{\partial v} I_{-v}(x) - I_v(x)\right]}{2\,\pi\cos(\pi v)}$$

$$= \lim_{v\to\mu} \frac{1}{2\cos(\pi v)}\left[\frac{\partial}{\partial v} I_{-v}(x) - \frac{\partial}{\partial v} I_v(x)\right]$$

According to the property $\frac{\partial}{\partial v} I_{-v}(x) = -\frac{\partial}{\partial v} I_v(x)$, by replacing:

$$= \lim_{v\to\mu} \frac{1}{2\cos(\pi v)}\left[-\frac{\partial}{\partial v} I_v(x) - \frac{\partial}{\partial v} I_v(x)\right]$$

$$= \lim_{v\to\mu} \frac{1}{\cos(\pi v)}\left[-\frac{\partial}{\partial v} I_v(x)\right]$$

Now evaluating the limit for $\mu = 0$:

$$K_0(x) = -\left[\frac{\partial}{\partial v} I_v(x)\right]|_{v=0} \tag{19}$$

The modified Bessel function $I_v(x)$ is defined as [4]:

$$I_v(x) = \left(\frac{x}{2}\right)^v \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+1)}\left(\frac{x}{2}\right)^{2n} \tag{21}$$

By property of Gamma Function $\Gamma(n + 1) = n!$, replacing in Eq. (21) [5]:

$$I_0(x) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2}\left(\frac{x}{2}\right)^{2n} \tag{22}$$

Equation (16) satisfies the form of the Eq. (22). Deriving now $I_v(x)$ in function of v.

$$\frac{\partial}{\partial v} I_v(x) = \left(\frac{x}{2}\right)^v \left[\sum_{n=0}^{\infty}\left(\frac{1}{n!}\right)\left(\frac{x}{2}\right)^{2n}\left(-\big(\Gamma(n+v+1)\big)^{-2}\right)\psi(n+v+1)\Gamma(n+v+1)\right] + \left(\frac{x}{2}\right)\ln\left(\frac{x}{2}\right)\sum_{n=0}^{\infty}\frac{1}{n!\,\Gamma(n+v+1)}\left(\frac{x}{2}\right)^{2n}$$

Simplifying and factoring:

$$\frac{\partial}{\partial v} I_v(x) = \left(\frac{x}{2}\right)^v \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+v+1)}\left(\frac{x}{2}\right)^{2n}\left[\ln\left(\frac{x}{2}\right) - \psi(n+v+1)\right] \tag{23}$$

where $\psi(n+v+1) = \frac{\Gamma'(n+v+1)}{\Gamma(n+v+1)}$. Replacing Eq. (23) in Eq. (19):

$$K_0(x) = -\left[\left(\frac{x}{2}\right)^v \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+v+1)} \left(\frac{x}{2}\right)^{2n} \left[\ln\left(\frac{x}{2}\right) - \psi(n+v+1)\right]\right]|_{v=0} \tag{24}$$

By evaluation in *v = 0* and using the Gamma Function property *Γ (n + 1) = n!* [5]:

$$K_0(x) = -\left[\sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \left[\ln\left(\frac{x}{2}\right) - \psi(n+1)\right]\right] \tag{25}$$

$$K_0(x) = -\ln\left(\frac{x}{2}\right)\left(\sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{x}{2}\right)^{2n}\right) + \left(\sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{x}{2}\right)^{2n}\right)\psi(n+1) \tag{26}$$

By replacing on Eq. (22) in Eq. (26):

$$K_0(x) = -\ln\left(\frac{x}{2}\right) I_0(x) + I_0(x)\psi(n+1) \tag{27}$$

Now, it is defined $\psi(n+1) = -\gamma + 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ and for $\psi(0) = -\gamma$. It is possible to write the function $\psi(n+1)$ as:

$$\psi(n+1) = \psi(0) + \sum_{p=1}^{n} \frac{1}{p} = -\gamma + \sum_{p=1}^{n} \frac{1}{p} \tag{28}$$

with γ as a Euler constant. By replacing Eq. (28) in Eq. (27):

$$K_0(x) = -\left(\ln\left(\frac{x}{2}\right) + \gamma\right) I_0(x) + I_0(x)\left(-\gamma + \sum_{p=1}^{n} \frac{1}{p}\right) \tag{29}$$

Simplifying and writing the Eq. (29):

$$K_0(x) = -\left(\ln\left(\frac{x}{2}\right) + \gamma\right) I_0(x) + \left(\sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{x}{2}\right)^{2n}\right)\left(\sum_{p=1}^{n} \frac{1}{p}\right) \tag{30}$$

Using a change of variable $x = m\lambda$ and modified the Eq. (30):

$$K_0(m\lambda) = -\left(\ln\left(\frac{m\lambda}{2}\right) + \gamma\right) I_0(m\lambda) + \left(\sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{m\lambda}{2}\right)^{2n}\right)\left(\sum_{p=1}^{n} \frac{1}{p}\right) \tag{31}$$

Eq. (31) is a modified Bessel function of zero order and of the second kind [4]. The general solution for the differential heat transfer equation of an annular fin is:

$$\theta(\lambda) = C_1 I_0(m\lambda) + C_2 K_0(m\lambda) \tag{32}$$

where $I_0(m\lambda)$ y $K_0(m\lambda)$ are defined by Eq. (16) and Eq. (31), respectively. Evaluating the boundary conditions for the adiabatic fin: First boundary condition $\theta(\lambda_1) = \theta_B$:

$$\theta(\lambda_1) = \theta_B = C_1 I_0(m\lambda_1) + C_2 K_0(m\lambda_1) \tag{33}$$

The second boundary condition requires the derivate of Eq. (32):

$$\frac{\partial\theta}{\partial\lambda} = C_1 I'_0(m\lambda_1) + C_2 K'_0(m\lambda_1) \tag{34}$$

In the modified Bessel functions it is satisfied that $I'_0(x) = I_1(x)$ and $K'_0(x) = -K_1(x)$. The functions $I_1(x)$ and $K_1(x)$ are modified Bessel functions of first order and of first and second class respectively. By replacing in Eq. (34):

$$\frac{\partial\theta}{\partial\lambda} = C_1 I_1(m\lambda) + C_2 K_1(m\lambda) \tag{35}$$

Evaluating the second boundary condition $\frac{\partial\theta}{\partial\lambda}|_{\lambda=\lambda_2} = 0$ in the Eq. (35):

$$\frac{\partial\theta}{\partial\lambda}|_{\lambda=\lambda_2} = 0 = C_1 I_1(m\lambda_2) - C_2 K_1(m\lambda_2) \tag{36}$$

The following system of equations is obtained:

$$C_1 I_0(m\lambda_1) + C_2 K_0(m\lambda_1) = \theta_B \tag{37}$$

$$C_1 I_1(m\lambda_2) - C_2 K_1(m\lambda_2) \tag{38}$$

Solving for C_1 in Eq. (38):

$$C_1 = \frac{C_2 K_1(m\lambda_2)}{I_1(m\lambda_2)} \tag{39}$$

By replacing Eq. (39) in Eq. (37):

$$\frac{C_2 K_1(m\lambda_2)}{I_1(m\lambda_2)} I_0(m\lambda_1) + C_2 K_0(m\lambda_1) = \theta_B \tag{40}$$

$$\frac{C_2K_1(m\lambda_2)I_0(m\lambda_1)+C_2K_0(m\lambda_1)I_1(m\lambda_2)}{I_1(m\lambda_2)}=\theta_B \tag{41}$$

Solving to C_2 from Eq. (41):

$$C_2=\frac{\theta_B I_1(m\lambda_2)}{K_1(m\lambda_2)I_0(m\lambda_1)+K_0(m\lambda_1)I_1(m\lambda_2)} \tag{42}$$

By replacing of Eq. (42) in Eq. (39):

$$C_1=\left(\frac{\theta_B I_1(m\lambda_2)}{K_1(m\lambda_2)I_0(m\lambda_1)+K_0(m\lambda_1)I_1(m\lambda_2)}\right)\frac{K_1(m\lambda_2)}{I_1(m\lambda_2)} \tag{43}$$

$$C_1=\left(\frac{\theta_B K_1(m\lambda_2)}{K_1(m\lambda_2)I_0(m\lambda_1)+K_0(m\lambda_1)I_1(m\lambda_2)}\right) \tag{44}$$

Using C_1 y C_2 in the general solution of the modified Bessel differential equation:

$$\theta(\lambda)=\left(\frac{\theta_B K_1(m\lambda_2)I_0(m\lambda)}{K_1(m\lambda_2)I_0(m\lambda_1)+K_0(m\lambda_1)I_1(m\lambda_2)}\right)+\left(\frac{\theta_B I_1(m\lambda_2)K_0(m\lambda)}{K_1(m\lambda_2)I_0(m\lambda_1)+K_0(m\lambda_1)I_1(m\lambda_2)}\right) \tag{45}$$

Factorizing θ_B and simplifying:

$$\frac{\theta(\lambda)}{\theta_B}=\left(\frac{K_1(m\lambda_2)I_0(m\lambda_1)+I_1(m\lambda_2)K_0(m\lambda)}{K_1(m\lambda_2)I_0(m\lambda_1)+K_0(m\lambda_1)I_1(m\lambda_2)}\right) \tag{46}$$

Eq. (46) corresponds to the solution to the differential heat transfer equation of an annular adiabatic fin [1]. The problem is demonstrated.

3. Conclusions

In this work, the mathematical development for obtaining the solution of the Bessel Equation applied to the description of heat transfer processes in adiabatic extended surfaces was presented. This intellectual effort serves as support to students of chemical and mechanical engineering and other related engineering careers, in the analysis and dimensioning of thermal equipment.

Authors' contributions
All contributions in the manuscript are from C. Nchikou.

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Not manifested

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